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MONOMORPHISMS AND EPIMORPHISMS IN HOMOTOPY THEORY

BY

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To the memory of my brother-in-law, Professor Jacob Feinstein

ABSTRACT

A natural question to ask in any category $\mathscr{C}$ is whether a morphism $f: X \to Y$ in $\mathscr{C}$ which is simultaneously mono and epi is actually an equivalence. In this paper, we study this question for the category $\mathscr{H}$ whose objects are pointed, path-connected CW-spaces and whose morphisms are pointed homotopy classes of maps. We also continue our study of Hopfian and co-Hopfian objects of $\mathscr{H}$ initiated in a recent joint paper with Peter Hilton.

In this paper, we pursue the themes discussed in an earlier, similarly titled paper [4].

Our first aim is to give an improvement of theorem 3 of [4], as follows.

THEOREM 1.1. *Let $e: X \to X$ be an epimorphism in $\mathscr{H}$, the pointed homotopy category of path-connected CW-spaces. If the integral homology groups H_nX, $n \geqq 1$, are Hopfian groups, then e is a homology equivalence.*

Theorem 3 of [4], which differs from Theorem 1.1 in that it requires the H_nX, $n \geqq 1$, to be finitely generated, is an immediate consequence of Theorem 1.1 since finitely generated abelian groups are certainly Hopfian. However, there are many nonfinitely generated abelian Hopfian groups; the P-localized integers $\boldsymbol{Z}_P$, as well as the P-adic integers $\hat{\boldsymbol{Z}}_P$, P a family of primes, provide simple such examples but there are also other, more exotic, examples. (See [1] for a brief survey.)

From Theorem 1.1, we deduce (cf. [4, corollary 4])

COROLLARY 1.1. *If X is a nilpotent space whose integral homology groups H_nX, $n \geqq 1$, are Hopfian groups, then X is a Hopfian object of $\mathscr{H}$.*

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The remaining results of [4, §2] may be similarly improved. We give the statements and remark that the proofs in [4] for the corresponding weaker versions go over verbatim to the present context.

THEOREM 1.2. *Let $f: X \to Y$, $g: Y \to X$ be epimorphisms in $\mathcal{H}$ and suppose the integral homology groups H_nX, $n \geqq 1$, are Hopfian groups. Then the integral homology groups H_nY, $n \geqq 1$, are Hopfian groups and f and g are homology equivalences.*

COROLLARY 1.2. *If, in addition, X and Y are nilpotent, then f and g are homotopy equivalences.*

It is natural to ask for dualizations of the results stated above, just as in [4, §3]. Such dualizations do, in fact, exist but are eminently uninteresting (they are practically tautologies). It may be noted that the only finitely generated abelian groups which are co-Hopfian are the finite abelian groups so that these trivial dualizations do *not* constitute generalizations of the results of [4, §3].

A generalization of corollary 4 of [4] of a different sort from Corollary 1.1 is given in [6]. There, it is shown that under suitable finiteness conditions on X, an epimorphism $e: X \to X$ must be a monomorphism in a certain weak sense, but one which is nevertheless sufficiently strong to allow the method of proof of [4, theorem 7] to yield the conclusion that e is a homotopy equivalence — we do not elaborate the details here. The following problem then naturally suggests itself: given a morphism $f: X \to Y$ in $\mathcal{H}$ which is simultaneously mono and epi, find general conditions for f to be a homotopy equivalence. By using a variant of the techniques of [4], we derive the following

THEOREM 2.1. *Let $f: X \to Y$ be both a monomorphism and an epimorphism in $\mathcal{H}$. If f is a nilpotent map (in the sense of [5, p. 67]) and if either H_nf or π_nf[†] is finitely generated for all $n \geqq 3$, then f is a homotopy equivalence.*

Appealing to [5, proposition 2.13, p. 67] we then obtain

COROLLARY 2.1. *Let X and Y be nilpotent spaces of finite type. If $f: X \to Y$ is both a monomorphism and an epimorphism in $\mathcal{H}$, then f is a homotopy equivalence.*

The proof of Theorem 1.1 will be carried out in §1; that of Theorem 2.1, together with discussion of a number of examples comparing Theorem 2.1 with the results in [4], in §2.

[†] We may think of H_nf as H_n (cofiber of f) and π_nf as π_{n-1} (fiber of f).

1. Proof of Theorem 1.1

The key point in improving [4, theorem 3] so as to obtain Theorem 1.1 is to study directly the induced map $e_*: H_*X \to H_*X$ on integral homology and to bypass consideration of the induced maps $e_*: H_*(X; Z/p) \to H_*(X; Z/p)$ on mod p homology, p an arbitrary prime. The following general proposition on epimorphisms $f: X \to Y$ in $\mathscr{H}$ provides the necessary information.

PROPOSITION 1.1. *Let $f: X \to Y$ be an epimorphism in $\mathscr{H}$. If, for some $m \geqq 0$, $f_*: H_mX \cong H_mY$, then $f_*: H_{m+1}X \twoheadrightarrow H_{m+1}Y$.*

PROOF. For any $k \geqq 0$ and any (constant) coefficient group G, we have $f^*: H^k(Y; G) \rightarrowtail H^k(X; G)$. The Universal Coefficient Theorem provides a map of short exact sequences

$$\begin{array}{ccccc} \mathrm{Ext}(H_mY, G) & \rightarrowtail & H^{m+1}(Y; G) & \twoheadrightarrow & \mathrm{Hom}(H_{m+1}Y, G) \\ \cong \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ \mathrm{Ext}(H_mX, G) & \rightarrowtail & H^{m+1}(X; G) & \twoheadrightarrow & \mathrm{Hom}(H_{m+1}X, G). \end{array}$$

It follows that $f^*: \mathrm{Hom}(H_{m+1}Y, G) \rightarrowtail \mathrm{Hom}(H_{m+1}X, G)$, from which we find $f_*: H_{m+1}X \twoheadrightarrow H_{m+1}Y$, as desired.

In light of [4, proposition 1], Proposition 1.1 yields

COROLLARY 1.3. *Let $f: X \to Y$ be an epimorphism in $\mathscr{H}$. If X is r-connected, then Y is also r-connected; further, $f_*: \pi_{r+1}X \twoheadrightarrow \pi_{r+1}Y$, $f_*: H_{r+1}X \twoheadrightarrow H_{r+1}Y$.*

REMARKS. (1) The hypothesis $f_*: H_mX \cong H_mY$ in Proposition 1.1 is needed only to ensure the surjectivity of $\mathrm{Ext}(H_mY, G) \xrightarrow{f^*} \mathrm{Ext}(H_mX, G)$. Any other hypothesis guaranteeing this, e.g. H_mX free abelian, would lead to the same conclusion.

(2) We do not know whether, in general, an epimorphism $f: X \twoheadrightarrow Y$ in $\mathscr{H}$ induces a surjection $f_*: H_nX \twoheadrightarrow H_nY$ for all $n \geqq 0$. For the 'classical' epimorphisms constructed in [3, theorem 15.11, p. 180], this is always the case; indeed, if f is as in loc. cit., then $\Sigma f: \Sigma X \to \Sigma Y$ is, up to homotopy, a retraction.

(3) If, as in [4], we work with mod p homology, p a prime, the situation is simpler. The absence of the Ext term in the Universal Coefficient Theorem in that case leads to the conclusion that any epimorphism $f: X \to Y$ in $\mathscr{H}$ induces a surjection $f_*: H_n(X; Z/p) \twoheadrightarrow H_n(Y; Z/p)$ for all $n \geqq 0$.

We now prove Theorem 1.1. From Proposition 1.1 (or [4, proposition 1]), we infer that $f_*: H_1X \twoheadrightarrow H_1X$, hence, as H_1X is Hopfian, that $f_*: H_1X \cong H_1X$. We proceed inductively, assuming that $f_*: H_kX \cong H_kX$, $k < n$ $(n \geqq 2)$. From Proposition 1.1, we infer that $f_*: H_nX \twoheadrightarrow H_nX$ and again, as H_nX is Hopfian, that $f_*: H_nX \cong H_nX$. This completes the induction step and establishes the theorem.

2. Proof of Theorem 2.1

We show inductively that $\pi_n f = 0$ for all $n \geqq 1$. To begin, we invoke [4, proposition 1] to infer that $f_*: \pi_1 X \twoheadrightarrow \pi_1 Y$, that is, $\pi_1 f = 0$. As f is a monomorphism, we also have $f_*: \pi_1 X \rightarrowtail \pi_1 Y$ so that $f_*: \pi_1 X \cong \pi_1 Y$. Abelianizing, we find that $f_*: H_1 X \cong H_1 Y$ and hence, by Proposition 1.1, that $f_*: H_2 X \twoheadrightarrow H_2 Y$. It then follows that $H_2 f = 0$ and therefore, by the relative Hurewicz theorem, that $\bar{\pi}_2 f = 0$, where $\bar{\pi}_k f$ denotes the quotient group of $\pi_k f$ obtained by killing the action of $\pi_1 X$. By the nilpotency of f, we infer that $\pi_2 f = 0$ (compare [2]).

We now assume $n \geqq 3$ and $\pi_k f = 0$, $k < n$. To prove that $\pi_n f = 0$, we make use, as in [4], of homology and homotopy with mod p coefficients, p an arbitrary prime. Consider the mod p homotopy-homology ladder

$$\begin{array}{ccccccccc}
\pi_n(X; Z/p) & \overset{f_*}{\rightarrowtail} & \pi_n(Y; Z/p) & \to & \pi_n(f; Z/p) & \overset{0}{\to} & \pi_{n-1}(X; Z/p) & \overset{f_*}{\rightarrowtail} & \pi_{n-1}(Y; Z/p) \\
\downarrow & & \downarrow & & \twoheaddownarrow & & \downarrow & & \downarrow \\
H_n(X; Z/p) & \overset{f_*}{\twoheadrightarrow} & H_n(Y; Z/p) & \overset{0}{\to} & H_n(f; Z/p) & \to & H_{n-1}(X; Z/p) & \overset{f_*}{\twoheadrightarrow} & H_{n-1}(Y; Z/p)
\end{array}$$

each of whose terms is well defined since $n \geqq 3$. The indicated injections on the top row follow from the fact that f is a monomorphism, while the indicated surjections on the bottom row follow from the fact that f is an epimorphism (see §1, Remark (3)). The surjectivity of the middle vertical arrow follows from the Hurewicz isomorphism $\bar{\pi}_n f \cong H_n f$ together with the isomorphisms $\pi_n(f; Z/p) \cong \pi_n f \otimes Z/p$, $H_n(f; Z/p) \cong H_n f \otimes Z/p$ coming from the appropriate Universal Coefficient Theorems. A simple diagram chase shows that $H_n(f; Z/p) = 0$. Since $H_n f$ is finitely generated and p is arbitrary, it then follows that $H_n f = 0$. By once again appealing to the nilpotency of f, we infer that $\pi_n f = 0$, thus completing the induction step and with it the proof of the theorem.

The contrast between Theorem 2.1 and the theorems of [4] is perhaps worthy of closer examination. Observe that since, in Theorem 2.1, $f: X \to Y$ is *both* a monomorphism *and* an epimorphism, no finiteness condition needs to be imposed on X or on Y but rather on f. In [4], $f: X \to X$ is *either* a monomorphism *or* an epimorphism and a finiteness condition was imposed on X (though we have seen, in Theorem 1.1, that a weaker condition would suffice for

[4, theorem 3]). The following examples demonstrate that a finiteness condition on f alone would, in fact, be insufficient for the theorems of [4]. The first two examples are of non-Hopfian objects of $\mathscr{H}$.

EXAMPLE 2.1. Let Z be a noncontractible, 1-connected space of finite type and $\nabla : Z \vee Z \to Z$ the folding map; plainly, ∇ is an epimorphism but not a monomorphism. Now let X be the countably infinite wedge $Z \vee Z \vee \cdots$ and define a map $e : X \to X$ by

$$X = (Z \vee Z) \vee X \xrightarrow{\nabla \vee 1_X} Z \vee X = X.$$

It is easy to see that e is an epimorphism but not a monomorphism and that $H_n e$ is finitely generated for all $n \geqq 1$.

EXAMPLE 2.2. Let $f : U \to V$ be an epimorphism (as in [3, pp. 180–181]) with U and V 1-connected of finite type and with some $f_* : \pi_s U \to \pi_s V$ nonsurjective (e.g. the 'collapsing' map $S^r \times S^r \to S^{2r}$, $r \geqq 2$, $s = 2r$). Notice that f cannot be a monomorphism (for instance, since $f_* : \pi_r U \to \pi_r V$ is noninjective).[†] Now let

$$X_k = \begin{cases} U, & k \leqq 0, \\ V, & k > 0, \end{cases} \qquad X = \bigvee_{k=-\infty}^{\infty} X_k$$

and let $e : X \to X$ be the evident 'shift' map corresponding to the scheme

$$\cdots \longrightarrow X_{-2} \xrightarrow{1_U} X_{-1} \xrightarrow{1_U} X_0 \xrightarrow{f} X_1 \xrightarrow{1_V} X_2 \xrightarrow{1_V} X_3 \longrightarrow \cdots.$$

Again, e is an epimorphism but not a monomorphism and $H_n e$ is finitely generated for all $n \geqq 1$.

While the epimorphism $e : X \to X$ in Example 2.1 is, up to homotopy, a retraction, so that $e_* : \pi_n X \twoheadrightarrow \pi_n X$ for all $n \geqq 1$, the epimorphism $e : X \to X$ in Example 2.2 possesses the feature that $e_* : \pi_s X \to \pi_s X$ is not surjective. We do not know whether there is a non-Hopfian object X of $\mathscr{H}$ such that the homotopy groups $\pi_n X$, $n \geqq 1$, are Hopfian but any such X would necessarily admit an epimorphism $e : X \to X$ possessing the feature just described.

Examples 2.1 and 2.2 may be readily dualized to yield (1-connected) non-co-Hopfian objects of $\mathscr{H}$.

EXAMPLE 2.1*. Let Z be a noncontractible, 1-connected space of finite type and $\Delta : Z \to Z \times Z$ the diagonal map; plainly, Δ is a monomorphism but not an

[†] Or by Corollary 2.1!

epimorphism. Now let X be the countably infinite product $Z \times Z \times \cdots$ and define a map $m : X \to X$ by

$$X = Z \times X \xrightarrow{\Delta \times 1_X} (Z \times Z) \times X = X.$$

It is easy to see that m is a monomorphism but not an epimorphism and that $\pi_n m$ is finitely generated for all $n \geqq 1$.

EXAMPLE 2.2*. Let $f : U \to V$ be a monomorphism with U and V 1-connected of finite type and with some $f_* : H_s U \to H_s V$ noninjective (e.g. the Hopf map $S^3 \to S^2$, $s = 3$). As in Example 2.2, f cannot be an epimorphism. Now let

$$X_k = \begin{cases} U, & k \leqq 0, \\ V, & k > 0, \end{cases} \qquad X = \prod_{k=-\infty}^{\infty} X_k$$

and let $m : X \to X$ be the evident 'shift' map corresponding to the scheme

$$\cdots \longrightarrow X_{-2} \xrightarrow{1_u} X_{-1} \xrightarrow{1_u} X_0 \xrightarrow{f} X_1 \xrightarrow{1_v} X_2 \xrightarrow{1_v} X_3 \longrightarrow \cdots.$$

Again, m is a monomorphism but not an epimorphism and $\pi_n m$ is finitely generated for all $n \geqq 1$.

These examples fortify the impression that the finiteness hypothesis in Theorem 2.1 is rather mild. In spite of this, it should be noted that the hypothesis that f be a monomorphism and an epimorphism is used sparingly; all that is needed in this connection is that f should induce $f_* : \pi_1 X \cong \pi_1 Y$, $f_* : H_2 X \twoheadrightarrow H_2 Y$, $f_* : H_n(X : Z/p) \twoheadrightarrow H_n(Y; Z/p)$ and $f_* : \pi_n(X; Z/p) \rightarrowtail \pi_n(Y; Z/p)$, $n \geqq 3$, p an arbitrary prime. Now, if X is a 2-connected space of finite type, $Y = \hat{X}$ and $f = c : X \to \hat{X}$ is profinite completion, the conditions just listed are met (see [7]). Of course, unless all the homotopy groups of X happen to be finite groups, $H_n c$ is not finitely generated for all $n \geqq 1$ (and similarly for $\pi_n c$) and the conclusion of Theorem 2.1 fails for c. It is thus legitimate to wonder whether a more substantial utilization of the monomorphism–epimorphism hypothesis would allow relaxation, or even removal, of the finiteness hypothesis in Theorem 2.1.

To conclude, we remark that it seems certain that some sort of fundamental group hypothesis in Theorem 2.1 (as well as in [4, corollary 4]) is critical. However, it also seems quite difficult to construct appropriate counterexamples. Further study in this direction is undoubtedly warranted.

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